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LETTER TO THE EDITOR

Dilute Potts model, duality and site–bond percolation†

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Abstract. A number of results are obtained for Potts models on an arbitrary planar lattice. It is shown that the dual of a dilute Potts model is a graph-generating function and that the dual of a constrained dilute Potts model is an undiluted Potts model. It is also shown that the dilute Potts model generates a correlated site–bond percolation. For uncorrelated site–bond percolations our analysis determines the percolation threshold from a knowledge of the critical point of a Potts model. This generalises a recent result of Kondor who uses a star-triangle transformation to derive this relationship for the honeycomb and triangle lattices.

The duality relation for the Potts model was first reported by Potts (1952) some 30 years ago. Since that time the Potts duality has been reformulated and generalised in various different forms (see, e.g., Kihara *et al* 1954, Mittag and Stephen 1971, Wu and Wang 1976, Essam 1979, Burkhardt 1979, Kasai *et al* 1980). A common feature of these relations is that they connect Potts models of similar types of interactions.

In this Letter we report on an extension of the Potts duality which is different from those previously reported. This is a duality for the dilute Potts model which has been of recent theoretical interest (Nienhuis *et al* 1979, 1980, Wu 1980). We shall show that the dual of the dilute Potts model is a graph-generating function and that the dual of a constrained dilute model is a regular (undiluted) Potts model. We also show that the dilute Potts model generates a *correlated* site–bond percolation, while the constrained model generates an uncorrelated percolation. This latter fact enables us to relate the percolation threshold of a site–bond percolation with the critical point of a corresponding Potts model. This relation, which is valid for any planar lattice, generalises a recent result of Kondor (1980) who obtained the same relation for the honeycomb and triangular lattices using a star-triangle transformation.

Consider a dilute q-state Potts model on a planar graph G of N sites and E edges. We write the partition function in the form of Nienhuis *et al* (1979):

$$Z^{(G)}(q; K', K^*, z_i) = \sum_{t_i=0}^{1} \sum_{\sigma_i=1}^{q} \left(\prod_E \exp\left[t_i t_j (K' + K^* \delta_{ij})\right] \right) \left(\prod_i z^{1-t_i} \right) \tag{1}$$

where $i = 1, 2, \ldots, N$, $\delta_{ij} = \delta_{Kr}(\sigma_i, \sigma_j)$, z_i^{-1} is the fugacity of the ith site, and the summation over σ_i is for $t_i = 1$ only.

Let D be the dual of G and consider 'subgraphs' D' on D generated as follows. Shade randomly chosen $f(D')$ faces of D and attach bonds to randomly chosen $b(D')$

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edges of D which do not border any shaded face. For example, if G is a 4×4 lattice, then D is a graph having 16 faces and 10 sites, including one site residing exterior to G. An example of D' is shown in figure 1.

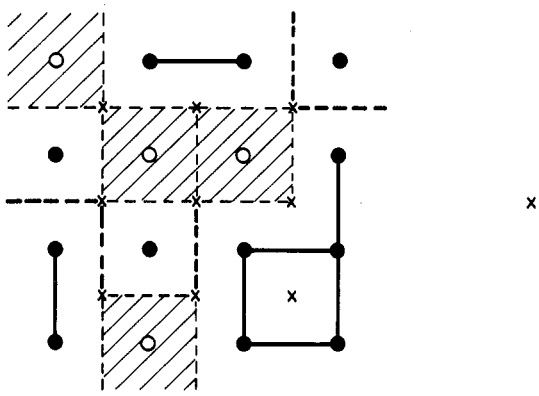

Figure 1. Example of a mapping between a subgraph G' on a 4×4 G lattice and a subgraph D' on D. The 16 sites of G are denoted by open and solid circles and the 10 sites of D are denoted by crosses. The single site of D residing exterior to G is connected to eight other sites of D.

In the configuration shown the open (solid) circles denote the vacant (occupied) sites in G' and the solid lines denote the attached bonds in G'. The broken lines denote the attached bonds in D'. Thus we have $N=16$, $E=24$, $f(G')=f(D')=4$, $b(G')=7$ (the solid lines), $h(G')=12$, $n(G')=6$, $c(G')=1$, $b(D')=5$ (the broken lines), $g(D')=1$, $n(D')=2$, $c(D')=5$, $\Sigma_i'\gamma_i=13$.

Two sites in D' belong to the same cluster if they are connected through a sequence of bonds and shaded faces. Let $n(D')$ be the number of clusters in D', including isolated sites, and $g(D')$ be the number of edges in D' bordering two shaded faces. We then define a graph-generating function on D as follows:

$$\Xi^{(D)}(q;v,t,u_i)=\sum_{D'\subseteq D}v^{b(D')}t^{g(D')}q^{n(D')}\prod_i{}'u_i \tag{2}$$

where u_i is a weight associated with the ith face of D, and the prime in Π'_i indicates that the product is taken over the $f(D')$ shaded faces only.

We now state the duality relation connecting $Z^{(G)}$ and $\Xi^{(D)}$ as a theorem.

Theorem 1.

$$Z^{(G)}(q;K',K^*,z_i)=q^{N-1}t^E\Xi^{(D)}(q;v,t,u_i) \tag{3}$$

with

$$\exp(K^*)=1+q/v,\qquad \exp(K')=vt,\qquad z_i=qu_it^{\gamma_i}, \tag{4}$$

where γ_i is the valence (coordination number) of the ith site of G.

For a large regular lattice G of coordination number γ, the boundary effect may be neglected and we can simply drop the subscript i throughout. But for finite graphs (and lattices) γ_i will be generally non-uniform, and it is essential to keep this distinction. In fact, because of the local property of the duality relation, theorem 1 and other results of this paper can be generalised further by letting K', K^* etc be edge-dependent. We shall not do so, however, for the sake of clarity in presentation.

Proof. The theorem follows straightforwardly from a graphical expansion of $Z^{(G)}$. Write

$$\exp(K^*\delta_{ij}) = 1 + [\exp(K^*) - 1]\delta_{ij} \tag{5}$$

in (1) and expand the product Π_E. After carrying out the summations over σ_i, we may write the partition function as a graph-generating function on G:

$$Z^{(G)}(q; K', K^*, z_i) = \sum_{G' \subseteq G} \exp[K' h(G)] [\exp(K^*) - 1]^{b(G')} q^{n(G')} \prod_i' z_i. \tag{6}$$

Here the subgraphs $G' \subseteq G$ are generated by randomly selecting $f(G')$ sites of G as being vacant (i.e. the remaining sites as being occupied), and attaching bonds to randomly chosen $b(G')$ of the $h(G')$ edges which connect occupied sites. Again, two occupied sites in G' belong to the same cluster if they are connected through a sequence of attached bonds and occupied sites. In (6), $n(G')$ is the number of clusters in G' (including isolated sites) and the product Π_i' is over the $f(G')$ vacant sites.

Now to each $G' \subseteq G$ a unique $D' \subseteq D$ can be constructed (and vice versa) as follows:

(i) shade those faces of D containing vacant sites of G';

(ii) attach bonds to the edges of D complementing those of G', i.e. attach a bond to an edge of D if the corresponding edge of G connects two occupied sites and is empty.

An example of this mapping for a 4×4 G is shown in figure 1. It is seen that (i) and (ii) generate precisely the subgraphs D' in (2). In particular, we have

$$f(D') = f(G'), \tag{7}$$

$$b(D') = h(G') - b(G'). \tag{8}$$

Also, since by construction each circuit of G' encircles a cluster of D' and vice versa, we have

$$n(D') = c(G') + 1 = n(G') + f(G') + b(G') - N + 1 \tag{9}$$

where $c(G')$ is the number of independent circuits in G' and the last equality in (9) is the Euler relation for G'. We have also the relation

$$h(G') = E - \sum_i' \gamma_i + g(D') \tag{10}$$

which relates $h(G')$, the number of edges in G' connecting occupied sites, to $g(D')$, the number of edges in G' connecting vacant sites. The summation in (10) is again taken over the $f(G') = f(D')$ vacant sites of G'.

It is now convenient to start from (2) and use (8)–(10) to eliminate $b(D')$, $g(D')$ and $n(D')$. The result is

$$\Xi^{(D)}(q; v, t, u_i) = q^{1-N} t^{-E} \sum_{G' \subseteq G} (vt)^{h(G')} \left(\frac{q}{v}\right)^{b(G')} q^{n(G')} \prod_i' (q u_i t_i^{\gamma_i}) \tag{11}$$

which reduces to (3) by comparing with (6).

To proceed further, we now establish the equivalence of a *constrained* dilute Potts model with a regular (undiluted) Potts model.

Consider a Potts model on D whose Hamiltonian $\mathcal{H}$ is given by

$$-\beta\mathcal{H} = K \sum_{nn} \delta_{kl} + \sum_i L_i \delta_i \tag{12}$$

where K is the nearest-neighbour interaction (on D) and L_i is a multi-spin interaction at the ith face of D with

$$\begin{aligned}&\delta_i = 1 \text{ if all spins surrounding the } i\text{th face of } D \text{ are in the same state,}\\ &\quad = 0 \text{ otherwise.}\end{aligned} \tag{13}$$

Denoting the partition function of (12) by $Z^{(D)}(q; K, L_i)$, we then have the following equivalence.

Theorem 2.

$$Z^{(G)}(q; K', K^*, z_i) = q^{N-1} \exp[-EK_Z(D)](q; K, L_i) \tag{14}$$

with

$$\exp(K^*) = [\exp(K) + q - 1]/[\exp(K) - 1], \qquad \exp(K') = 1 - \exp(-K),$$
$$z_i = q[\exp(L_i) - 1]. \tag{15}$$

Note that (15) implies the following constraint on the dilute model:

$$\exp(K') = q/[\exp(K^*) + q - 1]. \tag{16}$$

Theorem 2 can be established by writing in $Z^{(D)}$

$$\exp(L_i\delta_i) = 1 + [\exp(L_i) - 1]\delta_i, \qquad \exp(K\delta_{kl}) = 1 + [\exp(K) - 1]\delta_{kl}, \tag{17}$$

and expanding first the product $\Pi_i \exp(L_i\delta_i)$. Graphically representing the factor of $\exp(L_i) - 1$ by shading the ith face of D, it is then a simple matter to put $Z^{(D)}$ in the form of $\Xi^{(D)}$, and hence, by theorem 1, in the form of $Z^{(G)}$. Details are straightforward and will not be given.

Next we relate the dilute Potts model to a percolation problem. Consider a *correlated* site–bond percolation process on G, defined as follows. (i) The ith site of G is occupied with a probability s_i. (ii) Each edge of G is occupied with a probability p. (iii) The overall probability is enhanced by a factor x for *each* pair of neighbouring sites that are *both* occupied. (The percolation is uncorrelated when $x = 1$.) As usual, we consider two sites to be in the same cluster if they are connected through a sequence of occupied edges and sites. It follows that, in counting the number of clusters, the occupancies of those edges adjacent to vacant sites are irrelevant. Consequently, we may use precisely the subgraphs G' in (6) to describe the configurations arising in the percolation. In particular, we define, as in Kasteleyn and Fortuin (1969) (see also Wu (1978)), a random cluster generating function

$$\psi^{(G)}(q; p, x, s_i) = \sum_{G' \subseteq G} x^{h(G')} p^{b(G')} (1-p)^{h(G')-b(G')} q^{n(G')} \left(\prod_{\substack{\text{occupied}\\ \text{sites}}} s_i \right) \left(\prod_{\substack{\text{vacant}\\ \text{sites}}} (1-s_i) \right). \tag{18}$$

Here, the symbols $h(G')$, etc, have the same meaning as in (6). The usefulness of this generating function is illustrated by the fact that the mean number of clusters is

$$\langle n \rangle = \left(\frac{\partial}{\partial q} \ln \psi^{(G)}(q; p, x, s_i) \right)_{q=1}. \tag{19}$$

A direct comparison of (18) with (6) now yields the following identity:

$$\psi^{(G)}(q; p, x, s_i) = \left(\prod_{i=1}^{N} s_i\right) Z^{(G)}(q; K', K^*, z_i) \tag{20}$$

with

$$p = 1 - \exp(-K^*), \qquad x = \exp(K' + K^*), \qquad s_i = (1 + z_i)^{-1}. \tag{21}$$

As a result, the percolation threshold of the correlated site–bond percolation is identically the $q = 1$ limit of the critical point of the dilute Potts model (1).

Of particular interest is the uncorrelated ($x = 1$) site–bond percolation. For $x = 1$ it is seen that the condition (16) is always satisfied at $q = 1$, so that $\psi^{(G)}$ is further related to the Potts model (12) on D. The exact equivalence obtained from (15) and (21) now reads

$$s_i = \exp(-L_i), \qquad p = \exp(-K). \tag{22}$$

This is a very general result valid for any G and D. Namely, the threshold of a site–bond percolation is obtainable from a knowledge of the critical point of a related Potts model. For example, the threshold of the site–bond percolation on a square lattice is deduced from the critical point of the Potts model on a square lattice with nearest-neighbour interactions K and four-site interactions L_i. Similarly, the threshold of the site–bond percolation on the honeycomb lattice is obtainable from the Potts critical point for the triangular lattice with two- and three-site interactions. The latter fact has been established by Kondor (1980) from a star-triangle transformation of the percolation configurations. In fact, using the conjectured critical point (Wu 1979) for the Potts model, Kondor has deduced the criticality for this site–bond percolation. Unfortunately, it has been shown rigorously (Enting and Wu 1980, to be submitted for publication) that the Wu conjecture is inaccurate†. The exact site–bond percolation threshold for the honeycomb lattice remains unknown to this date.

For bipartite G, a dilute Potts model can be defined in which the vacancies can occur for only one of the two sublattices. For this restricted model we may simply take $K' = 0$ in (1) and $t = 1$ (since there are no neighbouring shaded faces in D') in (2). In addition to theorem 1, we also have the following:

$$Z_{\mathrm{B}}^{(G)}(q; K^*, z_i) = q^{N-1} v^{-E} \Xi_{\mathrm{B}}^{(D)}(q; v, u_i) \tag{23}$$

with

$$\exp(K^*) = 1 + q/v, \qquad z_i = q u_i v^{-\gamma_i}. \tag{24}$$

The subscript B here denotes the validity for bipartite G. The novel point here is that the dilute model is now completely equivalent to the regular Potts model (12) which has $L_i = 0$ in every other face. In addition to theorem 2, we also have the following equivalence:

$$Z_{\mathrm{B}}^{(G)}(q; K^*, z_i) = q^{N-1}[\exp(K) - 1]^{-E} Z_{\mathrm{B}}^{(D)}(q; K, L_i) \tag{25}$$

with

$$\exp(K^*) = [\exp(K) + q - 1]/[\exp(K) - 1], \qquad z_i = q[\exp(L_i) - 1][1 - \exp(-K)]^{-\gamma_i}. \tag{26}$$

† Vicsek and Kertész (1980) have also demonstrated that a Monte Carlo determination of the critical probability of the pure site percolation on the honeycomb lattice is inconsistent with the Wu conjecture.

Finally, both $Z_B^{(G)}$ and $Z_B^{(D)}$ can be further related to a random cluster generating function in the form of (18) with $x = 1$. In the percolation limit of $q = 1$, the result leads again to (23).

To summarise, we have established the equivalence between the dilute Potts model and a graph-generating function (theorem 1). We have also shown that the dual of a constrained dilute model is a regular (undiluted) model (theorem 2), thus completing a curious mapping of a $(q+1)$-state model to a model of q states. An immediate application of this result leads to the conclusion that a certain constrained Capel–Blume–Emery–Griffiths spin-1 model (Capel 1966b, Blume *et al* 1971) is reducible to a spin-$\frac{1}{2}$ Ising model. This fact, which differs from Capel (1966a) and Griffiths (1967), appears to have escaped previous attention. We have also established the connection between the dilute model and the site–bond percolation. In particular, we have shown that the percolation threshold of a site–bond percolation on any planar lattice G is obtainable from the critical point of a related Potts model on D.

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